Utility Functions
Case Study - Insurance Premiums

Gary Schurman MBE, CFA

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In this white paper we will use the exponential utility function to price an insurance polity. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

Imagine that we work for an insurance company that wants to underwrite an insurance policy to insure the following potential loss...

Table 1: Model Assumptions

Symbol	Description	Value
W	Current wealth of insured (\$)	1,000,000
Γ	Potential loss as a percent of wealth (%)	40.00
λ	Risk aversion coefficient (#)	3.00
T	Insurance policy term in years (#)	2.00
ω	Probability of loss over policy term (%)	2.50
μ	Risk-free interest rate (%)	4.25

Questions:
1. What is the insurance premium that this customer would pay?
2. Prove your answer to question one above.

The Exponential Utility Function

We have two possible scenarios...

$$\text{Scenario 1: Loss event does not occur ...and... Scenario 2: Loss event does occur} \tag{1}$$

We will define the variable W_i to be scaled wealth via the i'th scenario. The equation for scaled wealth is...

$$W_i = \frac{\text{Future wealth via the i'th scenario}}{\text{Current wealth}} \tag{2}$$

Using Equations (1) and (2) above and the data in Table 1 above, scenario scaled wealth for our case study is...

$$W_1 = \frac{1,000,000}{1,000,000} = 1.00 \text{ (No loss) ...and... } W_2 = \frac{1,000,000 \times (1 - 0.40)}{1,000,000} = 0.60 \text{ (Loss)} \tag{3}$$

We will define the variable p_i to be the probability of realizing the i'th scenario in Equation (1) above. Using Equation (2) above, the equation for expected scaled wealth is...

$$\mathbb{E}\Bigg[\text{Scaled wealth}\Bigg] = \sum_{i=1}^{n} p_i \, W_i \text{ ...where... } \sum_{i=1}^{n} p_i = 1 \tag{4}$$

Using Equations (3) and (4) above and the data in Table 1 above, expected scaled wealth at time T for our case study is...

$$\mathbb{E}\Bigg[\text{Scaled wealth}\Bigg] = 1.00 \times (1 - 0.025) + 0.60 \times 0.025 = 0.9900 \tag{5}$$

We will define the variable α to be a scalar whose value is greater than zero. Using Equation (2) above, the equations for the exponential untility function at its derivatives with respect to the scalar α are... [2]

$$U(W_i) = 1 - \text{Exp}\Big\{ - \alpha\, W_i \Big\} \text{ ...where... } U'(W_i) = \alpha\,\text{Exp}\Big\{ - \alpha\, W_i \Big\} \text{ ...and... } U''(W_i) = -\alpha^2\,\text{Exp}\Big\{ - \alpha\, W_i \Big\} \tag{6}$$

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equation (6) above and the data in Table 1 above, the equation for this measure of risk aversion is... [2]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = -\frac{-\alpha^2\,\text{Exp}\{-\alpha\, W_i\}}{\alpha\,\text{Exp}\{-\alpha\, W_i\}} = \alpha \text{ ...such that... } \alpha = \lambda = 3.00 \tag{7}$$

Using Equations (6) and (7) above, the equation for the expected utility of scaled wealth is... [2]

$$\mathbb{E}\Big[U(\text{Scaled wealth})\Big] = \sum_{i=1}^{n} p_i\, U(W_i) = \sum_{i=1}^{n} p_i \bigg(1 - \text{Exp}\Big\{ - \alpha\, W_i \Big\}\bigg) = 1 - \sum_{i=1}^{n} p_i\,\text{Exp}\Big\{ - \alpha\, W_i \Big\} \tag{8}$$

Using Equation (8) above and the data in Table 1 above, expected scaled utility at time T for our case study is...

$$\mathbb{E}\Big[U(\text{Scaled wealth})\Big] = 1 - \bigg((1 - 0.025) \times \text{Exp}\Big\{ - 3.00 \times 1.00 \Big\} + 0.025 \times \text{Exp}\Big\{ - 3.00 \times 0.60 \Big\}\bigg) = 0.9473 \tag{9}$$

Given that the insurance customer is risk-averse, we can make the following statement...

$$\mathbb{E}\Big[U(\text{Scaled wealth})\Big] < \mathbb{E}\Big[\text{Scaled wealth}\Big] \tag{10}$$

Insuring The Potential Loss

We will define the variable CE to be the value of the certainty equivalent at time T. The dollar value of the certainty equivalent is such that the utility of the certainty equivalent is equal to the utility of expected wealth. This statement in equation form is... [1]

$$U(CE) = \mathbb{E}\Big[U(W)\Big] \tag{11}$$

We will define the certainty equivalent to be wealth minus

$$CE = \frac{W - P}{W} \tag{12}$$

Using Equations (8) and (12) above, we can rewrite Equation (11) above as...

$$1 - \text{Exp}\Big\{ - \alpha\,\frac{W - P}{W} \Big\} = \mathbb{E}\Big[U(W)\Big] \tag{13}$$

Using Appendix Equation (20) above, the solution to Equation (13) above is...

$$P = W\,\bigg[1 + \ln\bigg(1 - \mathbb{E}\Big[U(W)\Big]\bigg)\,\alpha^{-1}\bigg] \tag{14}$$

We will define the variable μ to be the risk-free interest rate. Using the equations above, the insurance premium paid by the insured at time zero is...

$$\text{Insurance premium paid} = P\,(1 + \mu)^{-T} \tag{15}$$

The Answers To Our Hypothetical Problem

1. What is the insurance premium that this customer would pay?

Using Equations (7), (9) and (14) above, the certainty equivalent at time T is...

$$P = 1{,}000{,}000 \times \bigg[1 + \ln\bigg(1 - 0.9473\bigg) \times 3.00^{-1}\bigg] = 18{,}950 \tag{16}$$

Using Equations (15) and (16) above, the answer to the question is...

$$\text{Insurance premium paid} = 18,950 \times (1 + 0.0425)^{-2} = 17,400 \tag{17}$$

2. Prove your answer to question one above.

Using Equations (6), (7), (12) and (16) above, the utility of the certainty equivalent is...

$$U(CE) = 1 - \text{Exp}\left\{ -3.00 \times \frac{1,000,000 - 18,950}{1,000,000} \right\} = 0.9473 \tag{18}$$

Using Equation (9) above, the utility of wealth at time T is...

$$\mathbb{E}\bigg[U(\text{Scaled wealth})\bigg] = 0.9473 \tag{19}$$

Since the results of Equations (18) and (19) are the same, the premium calculated in Equation (17) above is correct.

Appendix

A. We want to solve Equation (14) above for the variable P (insurance premium)...

$$\begin{aligned}
1 - \text{Exp}\left\{ -\alpha\, \frac{W-P}{W} \right\} &= \mathbb{E}\bigg[U(W)\bigg] \\
\text{Exp}\left\{ -\alpha\, \frac{W-P}{W} \right\} &= 1 - \mathbb{E}\bigg[U(W)\bigg] \\
-\alpha\, \frac{W-P}{W} &= \ln\left(1 - \mathbb{E}\bigg[U(W)\bigg]\right) \\
1 - \frac{P}{W} &= -\ln\left(1 - \mathbb{E}\bigg[U(W)\bigg]\right)\alpha^{-1} \\
\frac{P}{W} &= 1 + \ln\left(1 - \mathbb{E}\bigg[U(W)\bigg]\right)\alpha^{-1} \\
P &= W\left[1 + \ln\left(1 - \mathbb{E}\bigg[U(W)\bigg]\right)\alpha^{-1}\right]
\end{aligned} \tag{20}$$

References

[1] Gary Schurman, *Introduction To Utility Funtions,* October, 2023.

[2] Gary Schurman, *The Exponential Utility Funtion,* October, 2023.